

## Suggested Solution to Assignment 1

1. (a) Since  $27 = 11011_2$  and  $17 = 10001_2$

$$\begin{array}{r} \text{Now } 11011_2 \\ \oplus 10001_2 \\ \hline 1010_2 \end{array}$$

Thus  $27 \oplus 17 = 1010_2 = 10$

(b) It follows from  $x \oplus 38 = 25$  that

$$\begin{aligned} x &= x \oplus 38 \oplus 38 \\ &= 25 \oplus 38 \\ &= 11001_2 + 100110_2 \\ &= 111111_2 \\ &= 63 \end{aligned}$$

(c) If  $x \oplus y \oplus z = 0$ , then

$$x \oplus y = x \oplus y \oplus z \oplus z = 0 \oplus z = z$$

Thus  $x = x \oplus y \oplus y = z \oplus y = y \oplus z$

2. (a) Since  $29 = 11101_2$ ,  $20 = 10100_2$  and  $15 = 1111_2$

$$\begin{array}{r} \text{Now } 11101_2 \\ 10100_2 \\ \oplus 1111_2 \\ \hline 110_2 \end{array}$$

Thus  $29 \oplus 20 \oplus 15 = 110_2 = 6$

(b) We have  $29 \oplus 20 \oplus 15 = 11101_2 \oplus 10100_2 \oplus 1111_2 = 110_2 = 6$

By looking at the leftmost column, the winning move is to reach  $11011_2$  from  $11101_2$  or  $10010_2$  from  $10100_2$  or  $1001_2$  from  $1111_2$ .

Thus, all winning moves are  $(27, 20, 15)$ ,  $(29, 18, 15)$  and  $(29, 20, 9)$ .

3. (a) Since  $12 = 1100_2$ ,  $19 = 10011_2$  and  $27 = 11011_2$

$$\text{We have } 12 \oplus 19 \oplus 27 = 1100_2 \oplus 10011_2 \oplus 11011_2 = 100_2 = 4$$

The only winning move is to take 4 from 12 since it is the only number 1 at the leftmost column.

Thus, the winning move is  $(8, 19, 27)$

(b) Since  $13 = 1101_2$ ,  $17 = 10001_2$ ,  $19 = 10011_2$  and  $23 = 10111_2$

We have  $13 \oplus 17 \oplus 19 \oplus 23 = 1101_2 \oplus 10001_2 \oplus 10011_2 \oplus 10111_2 = 11000_2 = 24$

By looking at the leftmost column, the winning moves are

$(13, 9, 19, 23)$ ,  $(13, 17, 11, 23)$  and  $(13, 17, 19, 15)$ .

4. (a)  $P = \{x \in \mathbb{N} : x \equiv 0, 2 \pmod{8}\}$  is the set of P-positions of the game.

(b) (i) The only terminal position is 0 and  $0 \in P$ .

(ii) Take any position  $p \in P$ . There are two cases: (1) when  $p \equiv 0 \pmod{8}$ ,

it can be moved to a position  $q$  where  $q \equiv 3, 4, 5, 7 \pmod{8}$ , which

shows that  $q \notin P$ ; (2) when  $p \equiv 2 \pmod{8}$ , it can be moved to a

position  $q$  where  $q \equiv 1, 5, 6, 7 \pmod{8}$ , which implies also that  $q \notin P$ .

In these two cases, any move from any position  $p \in P$  reaches a position  $q \notin P$ .

(iii) Suppose there are  $q$  chips in the pile where  $q \notin P$ . For  $q \equiv 1, 3, 5, 6, 7$

$\pmod{8}$ , the next player may remove 1, 3, 4, 5, 4, 5 chips such that the

remaining chips are congruent to 0, 0, 0, 0, 2, 2 modulo 8 respectively

and are positions in  $P$ . Therefore, for any position  $q \notin P$ , there exists a

move from  $q$  reaching a position  $p \in P$ .

Thus,  $P = \{x \in \mathbb{N} : x \equiv 0, 2 \pmod{8}\}$  is the set of P-positions.

So  $g(4) = 2$ ,  $g(18) = 0$ ,  $g(29) = 3$

5. (a)  $S = \{1, 2, 6\}$ , we list the values of  $g(x)$  below

$x$	0	1	2	3	4	5	6	7	8	9	10	11	12	...
$g(x)$	0	1	2	0	1	2	3	0	1	2	0	1	2	...

In fact, it can be shown that

$$g(x) = \begin{cases} 0 & , \text{ if } x \equiv 0, 3 \pmod{7} \\ 1 & , \text{ if } x \equiv 1, 4 \pmod{7} \\ 2 & , \text{ if } x \equiv 2, 5 \pmod{7} \\ 3 & , \text{ if } x \equiv 6 \pmod{7} \end{cases}$$

Since  $4 \equiv 4 \pmod{7}$ ,  $6 \equiv 6 \pmod{7}$ ,  $100 \equiv 2 \pmod{7}$

We have  $g(4) = 1$ ,  $g(6) = 3$  and  $g(100) = 2$ .

(b) By the hypothesis, the possible moves from 100 for the first player are

99, 98 and 94.

Since  $99 \equiv 1 \pmod{7}$ ,  $98 \equiv 0 \pmod{7}$ ,  $94 \equiv 3 \pmod{7}$ .

From (a), we have  $g(99)=1$ ,  $g(98)=0$  and  $g(94)=0$

Thus, 98 and 94 are P-positions, i.e. all winning moves are 98 and 94.

(c)  $P = \{x \in \mathbb{N} : x \equiv 0, 3 \pmod{7}\}$  is the set of P-positions of the game.

Pf. (i) The only terminal position is 0 and  $0 \in P$ .

(ii) Take any position  $p \in P$ . There are two cases: (1) when  $p \equiv 0 \pmod{7}$ ,

it can be moved to a position  $q$  where  $q \equiv 1, 2, 6 \pmod{7}$ , which shows

that  $q \notin P$ ; (2) when  $p \equiv 3 \pmod{7}$ , it can be moved to a position

$q$  where  $q \equiv 4, 5, 2 \pmod{7}$ , which also implies that  $q \notin P$ .

In these two cases, any move from any position  $p \in P$  reaches a position

$q \notin P$ .

(iii) Suppose there are  $q$  chips in the pile where  $q \notin P$ . For  $q \equiv 1, 4, 2, 5, 6 \pmod{7}$ , the next player may remove 1, 1, 2, 2, 6 chips respectively such that the remaining chips are congruent to 0, 3, 0, 3, 0 modulo 7 respectively.

and are positions in  $P$ . Therefore, for any position  $q \notin P$ , there exists

a move from  $q$  reaching a position  $p \in P$ .

Thus,  $P = \{x \in \mathbb{N} : x \equiv 0, 3 \pmod{7}\}$  is the set of P-positions.

7. (a) List the first few P-positions of the game:

$(0, 0), (1, 2), (3, 5), (4, 7), (6, 10), (8, 13), (9, 15), (11, 18), \dots$

Thus, for  $(6, 9)$ , the winning move is  $(4, 7)$ ; for  $(11, 15)$ , the winning moves

are  $(9, 15)$  and  $(6, 10)$ ; for  $(13, 20)$ , the winning moves are  $(13, 8)$ ,  $(12, 20)$

and  $(11, 18)$ .

(b) Let  $\varphi = \frac{1+\sqrt{5}}{2}$ .

(i) Suppose  $[n\varphi] = 100$ , then we can take  $n = 62$ . Thus  $[n\varphi] + n = 100 + 62 = 162$ .

Hence  $(x, y) = (100, 162)$ .

(ii) If we take  $[n\varphi] = 500$ , then  $n$  has no solutions. Thus we must choose  $n$  such that  $[n\varphi] + n = 500$ . We can take  $n = 191$ , then  $[n\varphi] = 309$  and  $[n\varphi] + n = 500$ .

Hence  $(x, y) = (500, 309)$ .

(iii)  $x - y = 999 \Rightarrow ([n\varphi] + n) - [n\varphi] = 999$ , so  $n = 999$ . Thus  $[n\varphi] = 1616$  and  $[n\varphi] + n = 2615$ .

Hence  $(x, y) = (2615, 1616)$ .

10. (a) According to the lectures, the Sprague-Grundy functions for 1-pile nim and subtraction game with  $S = \{1, 2, 3, 4, 5, 6\}$  are  $g_1(x) = x$  and  $g_2(x) \equiv x \pmod{7}$  respectively. It remains to find the Sprague-Grundy function  $g_3(x)$  for Game 3. We first list the values of  $g_3(x)$  as follows:

$x$	0	1	2	3	4	5	6	7	8	9	10	...
$g_3(x)$	0	1	1	0	2	0	3	0	4	0	5	...

In fact, it is not difficult to see that

$$g_3(x) = \begin{cases} 0 & , \text{ if } x \text{ is odd and } x \neq 1 \\ \frac{x}{2} & , \text{ if } x \text{ is even} \\ 1 & , \text{ if } x = 1 \end{cases}$$

Hence,  $g_1(14) = 14$ ,  $g_2(17) = 3$  and  $g_3(24) = \frac{24}{2} = 12$ .

(b) Since  $14 = 1110_2$ ,  $3 = 11_2$  and  $12 = 1100_2$

$$\begin{aligned} g(14, 17, 24) &= g_1(14) \oplus g_2(17) \oplus g_3(24) \\ &= 14 \oplus 3 \oplus 12 \\ &= 1110_2 \oplus 11_2 \oplus 1100_2 \\ &= 1_2 \\ &= 1 \end{aligned}$$

(c) Let the winning move of  $G$  from the position  $(14, 17, 24)$  be  $(x, y, z)$ .

We can make a move in exactly one of the game 1, 2, 3 (while the other two remain unchanged) such that  $g_1(x) = 111_2 = 15$ ,  $g_2(y) = 10_2 = 2$  or  $g_3(z) = 110_2 = 13$ . Note that since the number of chips is decreasing in succeeding moves, the case  $g_1(x) = 15$  is impossible.

For  $g_2(y) = 2$ , we may take  $y = 16$

For  $g_3(z) = 13$ , it is impossible too because the number of chips is decreasing in succeeding moves.

Hence, there is only one winning move from the position  $(14, 17, 24)$  to  $(14, 16, 24)$  (i.e. remove 1 chip in Game 2).

19. (a) According to the lectures, the Sprague-Grundy functions for 1-pile nim and subtraction game with  $S = \{1, 2, 3, 4, 5, 6, 7\}$  are  $g_1(x) = x$  and  $g_2(x) \equiv x \pmod{8}$  respectively. It remains to find the Sprague-Grundy function  $g_3(x)$  for Game 3.

We first list the values of  $g_3(x)$  as follows:

$x$	0	1	2	3	4	5	6	7	8	9	10	...
$g_3(x)$	0	1	2	1	0	1	2	1	0	1	2	

In fact, it is not difficult to see that

$$g_3(x) = \begin{cases} 0, & \text{if } x \equiv 0 \pmod{4} \\ 1, & \text{if } x \equiv 1, 3 \pmod{4} \\ 2, & \text{if } x \equiv 2 \pmod{4} \end{cases}$$

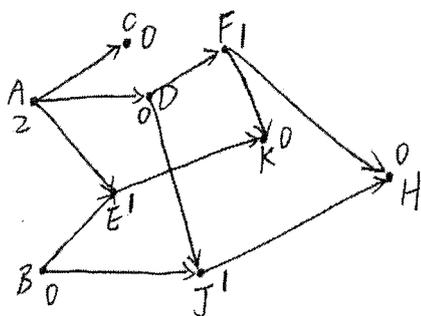
Hence,  $g_1(7) = 7$ ,  $g_2(14) = 6$ ,  $g_3(18) = 2$ .

(b) Since  $7 = 111_2$ ,  $6 = 110_2$ , and  $2 = 10_2$ :

$$\begin{aligned} g(7, 14, 18) &= g_1(7) \oplus g_2(14) \oplus g_3(18) \\ &= 7 \oplus 6 \oplus 2 \\ &= 111_2 \oplus 110_2 \oplus 10_2 \\ &= 11_2 \\ &= 3 \end{aligned}$$

- (c) Let the winning move of  $G$  from the position  $(7, 14, 18)$  be  $(x, y, z)$ .  
 We can make a move in exactly one of the game 1, 2, 3 such that  
 $g_1(x) = 10_2 = 4$ ,  $g_2(y) = 10_2 = 5$  or  $g_3(z) = 1_2 = 1$ .  
 For  $g_1(x) = 4$ , we may take  $x = 4$ .  
 For  $g_2(y) = 5$ , we may take  $y = 13$ .  
 For  $g_3(z) = 1$ , we may take  $z = 17$ .  
 Hence, all winning moves of  $G$  from the position  $(7, 14, 18)$  are  
 $(4, 14, 18)$ ,  $(7, 13, 18)$  and  $(7, 14, 17)$ .

12. (a)



- (b) By (a), all vertices which are at  $P$ -position but not at terminal position are:  
 $B$  and  $D$ .

- (c) (i) Since  $g_1(A) = 2$ ,  $g_2(B) = 0$ ,  $g_3(E) = 1$

$$\begin{aligned} \text{We have } g(A, B, E) &= g_1(A) \oplus g_2(B) \oplus g_3(E) \\ &= 2 \oplus 0 \oplus 1 \\ &= 10_2 \oplus 1_2 \\ &= 11_2 \\ &= 3 \end{aligned}$$

- (ii) Let the winning move from  $(A, B, E)$  be  $(x, y, z)$ . We can make a move  
 in exactly one of the three games such that  $g_1(x) = 0_2 = 1$ ,  $g_2(y) =$   
 $11_2 = 3$  or  $g_3(z) = 10_2 = 2$ .

Note that since the moves from  $A$  contain  $C, D, E$ ; from  $B$  contain  
 $E, J$ ; from  $E$  contains  $K$ .

For  $g_1(x) = 1$ , we may take  $x = E$ . For  $g_2(y) = 3$  or  $g_3(z) = 2$ , they are impossible.

Hence, the winning move from  $(A, B, E)$  is  $(E, B, E)$ .

13. (a) According to the lectures, the Sprague-Grundy function of the take-and-break game is given by

$$g(x) = \begin{cases} 4k+1, & \text{if } x = 4k+1 \\ 4k+2, & \text{if } x = 4k+2 \\ 4k+4, & \text{if } x = 4k+3 \\ 4k+3, & \text{if } x = 4k+4 \end{cases}$$

Since  $10 = 8+2$ ,  $11 = 8+3$  and  $12 = 8+4$

We have  $g(10) = 10$ ,  $g(11) = 12$ ,  $g(12) = 11$ .

$$\begin{aligned} (b) \quad g(10, 11, 12) &= g(10) \oplus g(11) \oplus g(12) \\ &= 10 \oplus 12 \oplus 11 \\ &= 1010_2 \oplus 1100_2 \oplus 1011_2 \\ &= 1101_2 \\ &= 13 \end{aligned}$$

Thus the next player has winning strategies moving to positions with Sprague-Grundy values either  $(111_2, 12, 11)$ ,  $(10, 1_2, 11)$  or  $(10, 12, 100_2)$ , i.e.  $(7, 12, 11)$ ,  $(10, 1, 11)$  or  $(10, 12, 6)$ .

For  $(7, 12, 11)$ , the player can remove 2 chips from the first pile to reach  $(8, 12, 11)$ .

For  $(10, 1, 11)$ , the player can remove 10 chips from the second pile to reach  $(10, 11, 11)$ .

For  $(10, 12, 6)$ , the player can remove 6 chips from the third pile to reach  $(10, 11, 6)$ .

Therefore, all winning moves from  $(10, 11, 12)$  are:

$(8, 11, 12)$ ,  $(10, 1, 12)$  and  $(10, 11, 6)$ .

Supplementary:

4. (c)  $S = \{1, 3, 4, 5\}$ , we list the values of  $g(x)$  below

$x$	0	1	2	3	4	5	6	7	8	9	10	11	12	...
$g(x)$	0	1	0	1	2	3	2	3	0	1	0	1	2	...

It can be shown that

$$g(x) = \begin{cases} 0, & \text{if } x \equiv 0, 2 \pmod{8} \\ 1, & \text{if } x \equiv 1, 3 \pmod{8} \\ 2, & \text{if } x \equiv 4, 6 \pmod{8} \\ 3, & \text{if } x \equiv 5, 7 \pmod{8} \end{cases}$$

Since  $4 \equiv 4 \pmod{8}$ ,  $18 \equiv 2 \pmod{8}$ ,  $29 \equiv 5 \pmod{8}$ ,

we have  $g(4) = 2$ ,  $g(18) = 0$  and  $g(29) = 3$ .

9: (a) According to the lectures, the set of P-positions of the ordinary nim game is

$$P_0 = \{ (\delta_1, \delta_3, \delta_5) : \delta_1 \oplus \delta_3 \oplus \delta_5 = 0 \}$$

Now we need to show  $P = \{ (\delta_1, \delta_2, \delta_3, \delta_4, \delta_5) : \delta_1 \oplus \delta_3 \oplus \delta_5 = 0 \}$  is the set of P-positions of the staircase nim game.

Check the 3 conditions for P-positions as follows.

(i) The only terminal position is  $(0, 0, 0, 0, 0)$  and  $(0, 0, 0, 0, 0) \in P$ .

(ii) Take any position  $(\delta_1, \delta_2, \delta_3, \delta_4, \delta_5) \in P$ . Then  $\delta_1 \oplus \delta_3 \oplus \delta_5 = 0$ . Suppose the position  $(\delta_1, \delta_2, \delta_3, \delta_4, \delta_5)$  is moved to  $(\delta'_1, \delta'_2, \delta'_3, \delta'_4, \delta'_5)$ . We know only one of  $\delta_1, \delta_3, \delta_5$  is changed. Without loss of generality, we may assume  $\delta'_1 < \delta_1$  or  $\delta'_1 > \delta_1$ ,  $\delta'_3 = \delta_3$ ,  $\delta'_5 = \delta_5$ .

Now if  $\delta'_1 \oplus \delta'_3 \oplus \delta'_5 = 0$ , then by cancellation law,

$$\delta'_1 \oplus \delta_3 \oplus \delta_5 = \delta_1 \oplus \delta_3 \oplus \delta_5 \Rightarrow \delta'_1 = \delta_1$$

which is a contradiction. Thus a position in  $P$  cannot move to a position in  $P$ .

(iii) Suppose  $(\delta_1, \delta_2, \delta_3, \delta_4, \delta_5) \in P$ . Then  $\delta_1 \oplus \delta_3 \oplus \delta_5 \neq 0$ . When calculating the nim sum using the binary expressions of  $\delta_1, \delta_3, \delta_5$ , look at the leftmost column with odd number of 1. Choose a pile that the corresponding binary digit of the number of chips in the pile is 1. Remove the chips from the pile for the ~~fifth~~ pile or remove the chips from the second or fourth pile to the first or third pile respectively so that all columns contain even number of 1 and the nim-sum of 3 numbers would become 0. Therefore any position not in  $P$  has a move to a position in  $P$ .

Thus,  $(\delta_1, \delta_2, \delta_3, \delta_4, \delta_5)$  is a P-position if and only if  $(\delta_1, \delta_3, \delta_5)$  is a P-position in ordinary nim.

(b) From (a), since  $4 = 100_2$ ,  $9 = 1001_2$ ,  $14 = 1110_2$ , we have

$$4 \oplus 9 \oplus 14 = 100_2 \oplus 1001_2 \oplus 1110_2 = 11_2 = 3$$

By looking at the leftmost column, the winning move <sup>to reach</sup> is  $11_2$  from  $100_2$ ,

$1010_2$  from  $1001_2$  or  $1101_2$  from  $1110_2$ .

Thus, all winning moves from the position  $(4, 6, 9, 11, 14)$  are:

$(7, 3, 9, 11, 14)$ ,  ~~$(4, 7, 8, 11, 14)$~~   $(4, 6, 10, 10, 14)$  and  $(4, 6, 9, 12, 13)$ .

6. (a) Note that the only terminal position is 0, 1.

By backwards induction, we have.

x	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
g(x)	0	0	1	1	2	0	3	1	2	0	0	1	1	2	0

x	15	16	17	18	19	20	...
g(x)	3	1	2	0	0	1	...

It is easy to see that g is periodic with period 9. Indeed,

$$g(x) = \begin{cases} 0, & \text{if } x \equiv 0, 1 \text{ or } 5 \pmod{9} \\ 1, & \text{if } x \equiv 2, 3 \text{ or } 7 \pmod{9} \\ 2, & \text{if } x \equiv 4 \text{ or } 8 \pmod{9} \\ 3, & \text{if } x \equiv 6 \pmod{9} \end{cases}$$

$$\Rightarrow g(6) = 3, \quad g(13) = 2, \quad g(100) = 0$$

(b) by the hypothesis, the possible moves from 13 are  $\{11, 10, 7\}$

$$g(11) = 1, \quad g(10) = 0, \quad g(7) = 1$$

$\Rightarrow 10$  is P-position i.e. all winning move for 13 is 10.

the possible moves from 34 are  $\{32, 31, 28\}$

$$g(28) = 0, \quad g(31) = 2, \quad g(32) = 1$$

28 is P-position, i.e. all winning move for 34 is 28.

(c) Claim: The set of P-positions is given by

$$P = \{k \in \mathbb{N} : k \equiv 0, 1 \text{ or } 5 \pmod{9}\}$$

proof: (1) terminal position  $k=0$  or  $1$  are in P.

(2) If  $k \equiv 0, 1$  or  $5 \pmod{9}$ , then we have  $k-2 \equiv 7, 8$  or  $4 \pmod{9}$ ,  $k-3 \equiv 6, 7$  or  $2 \pmod{9}$  and  $k-6 \equiv 3, 4$  or  $8 \pmod{9}$ .

Hence any position in P can only be removed to a position outside P.

(3) If  $k \not\equiv 0 \pmod{9}$ ,  $k \not\equiv 1 \pmod{9}$  and  $k \not\equiv 5 \pmod{9}$ ,  
 it is easy to see that  $k-2$ ,  $k-3$  and  $k-6$  is in  $P$ .

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8. (a) List the first few  $P$ -positions of the game:

$(0,0)$ ,  $(1,2)$ ,  $(3,5)$ ,  $(4,7)$ ,  $(6,10)$ ,  $(8,13)$ ,  $(9,15)$ ,  $(11,18)$ ,  
 $(12,20)$ ,  $(14,23)$ ,  $\dots \Rightarrow a=9$  (11,18)

$$(b) (b+60) - b = 60 \Rightarrow ([n\varphi] + n) - [n\varphi] = 60$$

$$\Rightarrow n = 60$$

$$\Rightarrow \lfloor 60 \times \frac{1+\sqrt{5}}{2} \rfloor = 97$$

$$\Rightarrow (b, b+60) = (97, 157)$$

(c) All winning moves for  $(15, 23)$  are  $(15, 9)$ ,  $(12, 20)$ ,  
 $(14, 23)$

As to  $(100, 160)$

$$1^\circ \lfloor [n\varphi] \rfloor = 100 \quad \varphi = \frac{1+\sqrt{5}}{2} \Rightarrow n = 62$$

$\Rightarrow (100, 162)$  It is impossible!

$$2^\circ \lfloor [n\varphi] \rfloor + n = 160, \quad \varphi = \frac{1+\sqrt{5}}{2} \Rightarrow 61 \leq n \leq 62$$

62 is impossible by  $1^\circ$  (reject)

If  $n = 61 \Rightarrow \lfloor [n\varphi] \rfloor + n = 159$  (reject!)

$$3^\circ 160 - 100 = 60 \Rightarrow n = 60 \Rightarrow \lfloor [n\varphi] \rfloor = 97$$

$$\Rightarrow 97 + 60 = 157$$

In all, all winning moves for  $(100, 160)$  is  
 $(97, 157)$

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174.  
a.

$\Rightarrow$  Denote  $\lfloor n\alpha \rfloor = k$ . then we have

$$0 < n\alpha - k < 1$$

$$k < n\alpha < k+1$$

$$\frac{k}{\alpha} < n < \frac{k+1}{\alpha}$$

Since  $\frac{k+1}{\alpha} > n$ , then  $\lfloor \frac{k}{\alpha} \rfloor \geq n-1$

Since  $\frac{k}{\alpha} < n$ , then  $\lfloor \frac{k}{\alpha} \rfloor \leq n-1$

Then  $\lfloor \frac{k}{\alpha} \rfloor = n-1$

Then  $\frac{k+1}{\alpha} - \lfloor \frac{k}{\alpha} \rfloor > 1$

$$\frac{k}{\alpha} - \lfloor \frac{k}{\alpha} \rfloor > 1 - \frac{1}{\alpha}$$

$$\left\{ \frac{k}{\alpha} \right\} > \frac{1}{\alpha}$$

$\Leftarrow$  Let  $n = \lfloor \frac{k}{\alpha} \rfloor + 1$ . Then  $n\alpha = \lfloor \frac{k}{\alpha} \rfloor \alpha + \alpha$   
 $= \frac{k}{\alpha} \cdot \alpha - \left\{ \frac{k}{\alpha} \right\} \alpha + \alpha$   
 $= k + \alpha \left( 1 - \left\{ \frac{k}{\alpha} \right\} \right)$

Then  $n\alpha > k + \alpha \cdot (1-1) = k$

$n\alpha < k + \alpha \cdot \left( 1 - \frac{1}{\alpha} \right) \leq k+1$

Thus  $\lfloor n\alpha \rfloor = k$

b.

Suppose there is no positive integer  $n$  s.t.  $\lfloor n\alpha \rfloor = k$

Then  $\left\{ \frac{k}{\alpha} \right\} < \frac{1}{\alpha}$

From  $k > \frac{k}{\alpha}$ , we have  $\left\{ k - \frac{k}{\alpha} \right\} = 1 - \left\{ \frac{k}{\alpha} \right\} > \frac{1}{\alpha}$

and  $\left\{ \frac{k}{\alpha} \right\} > \frac{1}{\alpha}$

Thus there exists positive integer  $n$  s.t.  $\lfloor n\beta \rfloor = k$   
For the other side, it's similar.

15.

$$b. \quad g(6, 13, 25) = 5 \oplus 11 = 101_2 \oplus 1011_2 = 1110_2 = 14_{10}$$

$5 \oplus 14$  is not achievable,

$$11 \oplus 14 = 5_{10}$$

Thus winning move is to  $(6, 13, 19)$

$$g(23, 56, 63) = 22 \oplus 6 = 10110_2 \oplus 110_2 = 10000_2 = 16_{10}$$

$$22 \oplus 16 = 6_{10}$$

$$6 \oplus 16 = 22_{10}$$

Thus winning moves are  $(7, 56, 63)$  and  $(23, 40, 63)$